Spread spectrum for imaging techniques in radio interferometry and strings detection

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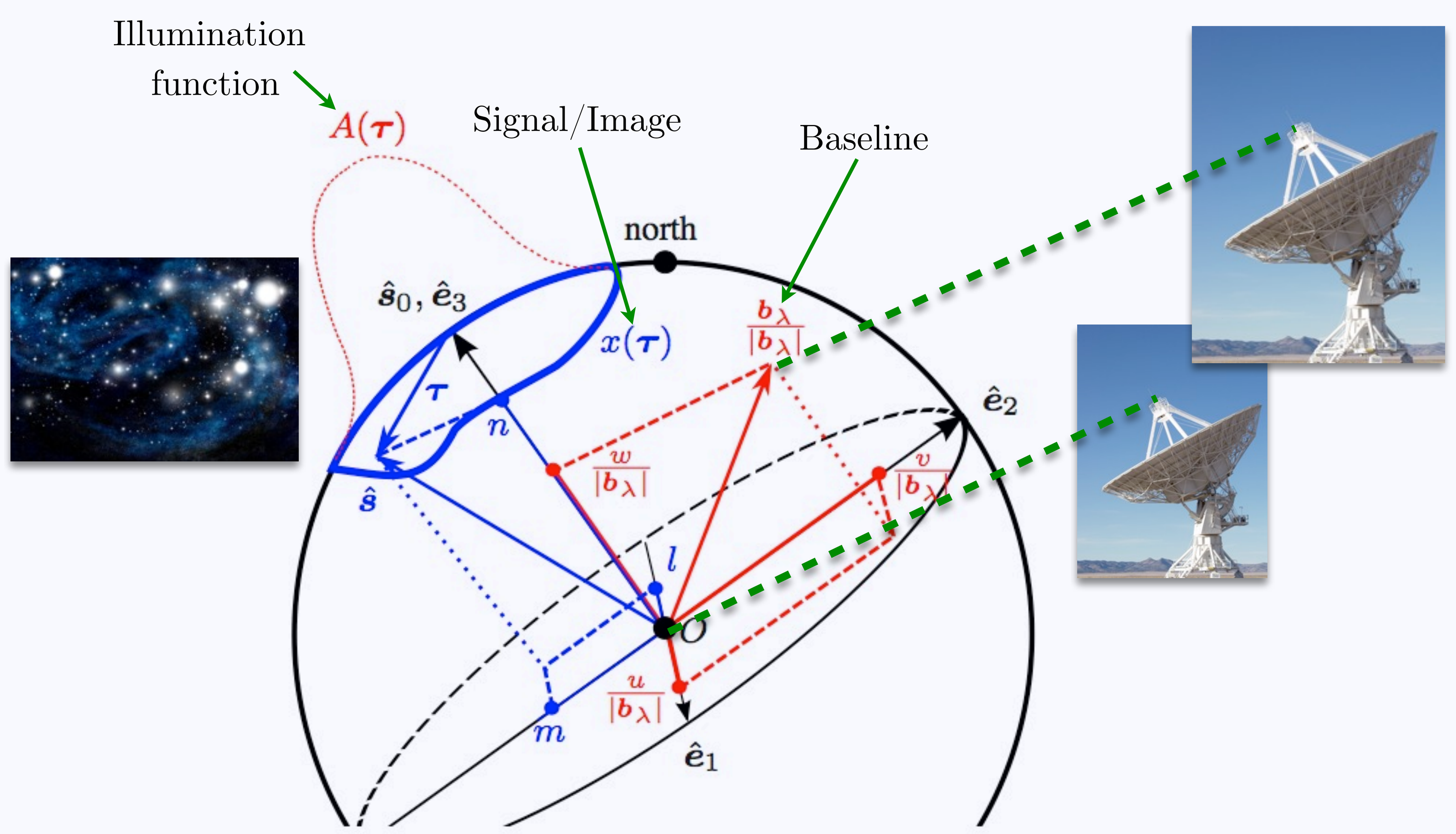
Radio interferometry:

Probing signals through incomplete Fourier measurements



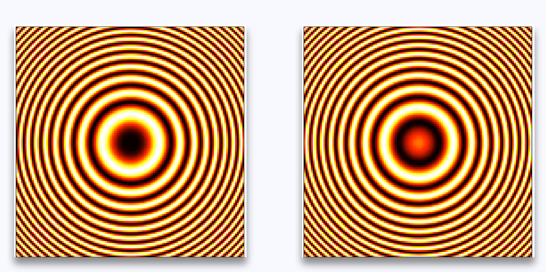
- Each telescope pair measures a visibility: $n(\boldsymbol{l}) = (1-|\boldsymbol{l}|^2)^{1/2}$

$$y(\boldsymbol{b}_\lambda) = y(\boldsymbol{u}, w) = \int_{D^2} Ax(\boldsymbol{l})\, e^{-2i\pi[\boldsymbol{u}\cdot\boldsymbol{l} + w(n(\boldsymbol{l})-1)]}\, n^{-1}(\boldsymbol{l})\, d^2\boldsymbol{l}.$$

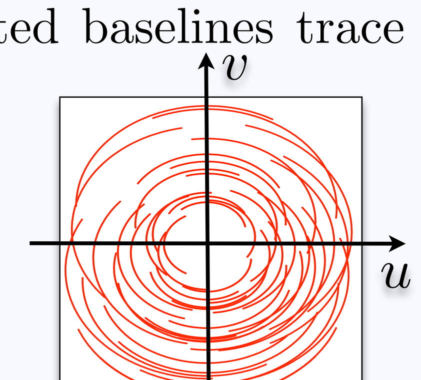
- For small field of view the signal x is planar.
- For non-negligible and constant baseline component w, a spread spectrum phenomenon appears:

$$y(\boldsymbol{u}) = \widehat{C^{(w)}Ax}(\boldsymbol{u}) = \left(\widehat{C^{(w)}} \star \widehat{Ax}\right)(\boldsymbol{u})$$

with $C^{(w)}(|\boldsymbol{l}|) = e^{i\pi w|\boldsymbol{l}|^2}$.



- Thanks to the Earth rotation, the projected baselines trace out arcs of ellipses on the Fourier plane.
  - The Fourier coverage is incomplete.
  - Visibilities are unavoidably affected by instrumental noise n.



- In the perspective of signal reconstruction, an ill-posed inverse problem has to be solved:

$$\boldsymbol{y} \equiv \Phi^{(w,t_0)}\boldsymbol{x} + \boldsymbol{n} \text{ with } \Phi^{(w,t_0)} \equiv \mathsf{MFC}^{(w)}\mathsf{A}^{(t_0)}.$$

Illumination function size; $\mathbb{C}^{M/2}$ *Visibilities*; $\mathbb{C}^{(M/2)\times N}$ *Sensing matrix*; $\mathbb{R}^N$ *Image*; $\mathbb{C}^{M/2}$ *Noise*; $\mathbb{R}^{(M/2)\times N}$ *Visibility mask*; $\mathbb{C}^{N\times N}$ *Fourier transform*; $\mathbb{C}^{N\times N}$; $\mathbb{R}^{N\times N}$

Thompson et al., 2004, "Interferometry ans Synthesis in Radio Astronomy",WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

Compressed sensing:

Beating Nyquist-Shannon theorem for sparse signals

- $\boldsymbol{x} \in \mathbb{R}^N$ is sparse or compressible in a basis $\Psi \in \mathbb{R}^{N\times N}$: $\boldsymbol{x} \equiv \Psi\boldsymbol{\alpha}$ with $\boldsymbol{\alpha} \in \mathbb{R}^N$ containing K non-zero or significant entries.
- $\boldsymbol{x}$ is probed by $M < N$ linear measurements, with i.i.d. Gaussian noise: $\boldsymbol{y} \equiv \Theta\boldsymbol{\alpha} + \boldsymbol{n} \in \mathbb{R}^M$ with $\Theta \equiv \Phi\Psi \in \mathbb{R}^{M\times N}$.
- The condition for accurate and stable recovery for random selection of Fourier measurements, i.e. $\Phi \equiv \mathsf{MF} \in \mathbb{R}^{M\times N}$, reads as:

$$K \leq \frac{cM}{N\mu^2(\mathsf{F},\Psi)\ln^4 N}, \text{ for coherence: } \mu(\mathsf{F},\Psi) \equiv \max_{1\leq i,j\leq N} |\widehat{\psi_j}(\boldsymbol{u}_i)|.$$

- The lower the coherence, the higher the sparsity recovered. The optimal limit is obtained for the Dirac basis Δ: $\lim_{N\to\infty} \mu(\mathsf{F},\Delta) = 0$.
- $\boldsymbol{x}$ is recovered by solving the Basis Pursuit minimization problem BP_ϵ:

$$\min_{\bar{\boldsymbol{\alpha}}\in\mathbb{R}^N} ||\bar{\boldsymbol{\alpha}}||_1 \text{ subject to } ||\boldsymbol{y} - \Theta\bar{\boldsymbol{\alpha}}||_2 \leq \epsilon.$$

Candès et al., 2006, IEEE Trans. Inf. Theory, 52, 5406

Chirp modulation and coherence:

Recovering optimal coherence whatever the sparsity basis

- For simplicity, we consider signals made up of Gaussian waveforms of size t: $\Psi \equiv \Gamma^{(t)}$.
- The coherence takes the general form:

$$\mu\left(\mathsf{FC}^{(w)}\mathsf{A}^{(t_0)}, \Gamma^{(t)}\right) = \frac{2tt_0}{t^2+t_0^2}\left[1 + \left(\frac{2\pi w t^2 t_0^2}{t^2+t_0^2}\right)^2\right]^{-\frac{1}{2}}.$$

- The power spectrum has a Gaussian shape of size $\propto 1/t$. The coherence decreases when t decreases, with the expected optimal limit:

$$\lim_{t\to 0} \mu\left(\mathsf{FC}^{(w)}\mathsf{A}^{(t_0)}, \Gamma^{(t)}\right) = 0 \text{ for all } w, t_0 \in \mathbb{R}_+.$$

- The spread spectrum phenomenon due to the chirp preserves the norm of the waveforms. The coherence decreases when the chirp rate increases, with the optimal limit obtained independently of t:

$$\lim_{w\to\infty} \mu\left(\mathsf{FC}^{(w)}\mathsf{A}^{(t_0)}, \Gamma^{(t)}\right) = 0 \text{ for all } t, t_0 \in \mathbb{R}_+.$$

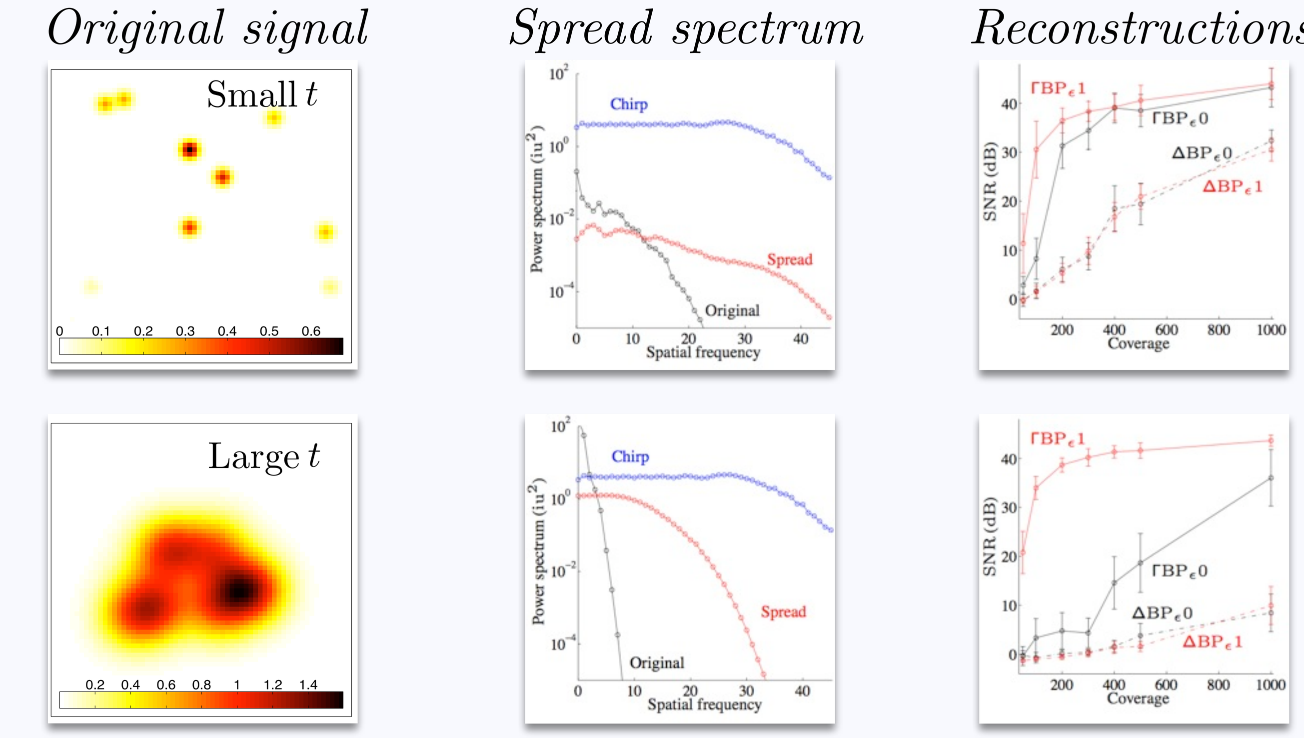
- This result suggests the universality of the spread spectrum phenomenon relative to the sparsity basis, in terms of achievable reconstruction quality.

Wiaux et al., 2009, Mon. Not. R. Astron. Soc, Submitted, arXiv:0907.0944v1

Spread spectrum universality:

Confirming theoretical results from simulations

- Signals are made up of 10 waveforms in $\Gamma^{(t)}$ for 2 values of t.
- Noisy visibilities are simulated for 2 values of w (0 and 1).
- The BP_ϵ problem is solved with 2 assumed sparsity dictionaries: for $\Gamma^{(t)}$ optimal sparsity or Δ for optimal coherence ($\equiv$ CLEAN).



1. $\Gamma\mathrm{BP}_\epsilon$ better than $\Delta\mathrm{BP}_\epsilon$: rather optimize sparsity than coherence!
2. $\Delta\mathrm{BP}_\epsilon 1$ equivalent to $\Delta\mathrm{BP}_\epsilon 0$ as μ is already optimal for $\Delta\mathrm{BP}_\epsilon 0$.
3. $\Gamma\mathrm{BP}_\epsilon 1$ better than $\Gamma\mathrm{BP}_\epsilon 0$ as μ is lower.
4. $\Gamma\mathrm{BP}_\epsilon 1$ independent of t: spread spectrum universality confirmed!

Wiaux et al., 2009, Mon. Not. R. Astron. Soc, Submitted, arXiv:0907.0944v1

Cosmic strings detection:

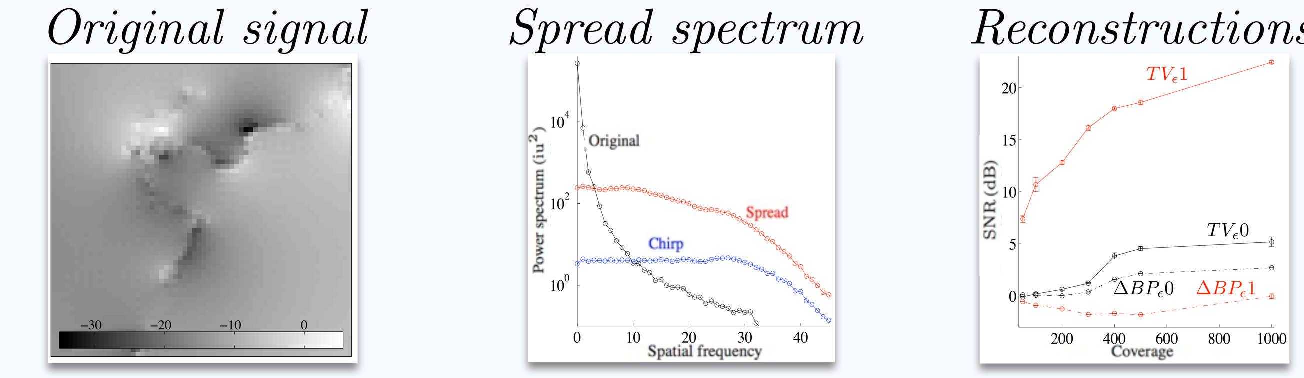
An example of application

- Cosmic strings imprint temperature steps in the CMB.
- The string signal exhibits sparse gradient and can be recovered by solving the Total Variation minimization problem TV_ϵ (Gaussian CMB discarded for the sake of illustration):

$$\min_{\bar{\boldsymbol{x}}\in\mathbb{R}^N} ||\bar{\boldsymbol{x}}||_{\mathrm{TV}} \text{ subject to } ||\boldsymbol{y} - \Psi\bar{\boldsymbol{x}}||_2 \leq \epsilon.$$

1-norm of gradient magnitude.

- Results are in line with previous considerations and conclusions:



Courtesy Fraisse et al.

Conclusion and future work

- Signal reconstruction quality is enhanced by a chirp modulation for non-negligible and constant baseline component w. It is independent of the sparsity basis for high enough w: spread spectrum universality!
- Results should be extended for signals on a wide field of view in the perspective of forthcoming radio interferometers such as the Square Kilometer Array (SKA).